Synchronization of the trajectory as a way to control the dynamics of a coupled system

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We introduce a scheme of controlling the dynamics of a deterministic system by coupling it to the dynamics of another similiar system. The controlled system synchronizes its dynamics with the control signal in the periodic as well as chaotic regimes. The method can be seen also as another way of controlling the chaotic behavior of a coupled system. [S1063-651X(97)15608-2]

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I. INTRODUCTION

Controlling the dynamics of a potentially chaotic system is presently a very active line of research in view of the applications in communication and system identification in different branches of science. Ott, Grebogi, and Yorke (OGY), in their seminal paper [1], showed how one can achieve chaotic control by using a feedback mechanism acting on the control parameters of the system. Small, carefully chosen, perturbations of the control parameters can drive the system to a fixed state or a periodic orbit. Since the introduction of that method, it has been used effectively to control the dynamics of many different theoretical and experimental systems [2,3]. The issue of control can be extended to the problem of the synchronization of coupled systems. For example, synchronized behavior has been studied in mean-field coupled Lorenz oscillators [4], diffusively coupled Lorenz systems [5], neural networks [6–8], and laser systems [9].

This paper investigates the properties of a coupled system where a target map is controlled by another map. The target map self-synchronizes its dynamics to that of the controlling map. The method described here is a generalization of the dynamic feedback control scheme [10–12] of driving the perturbed system to a fixed-point orbit. The controlling system is a map of the same kind as the target system. Before the control is initiated, both maps display quite different types of dynamics because their control parameters are different. In the extreme case the dynamics of the one system will be limited to a fixed point, while the second system will be fully chaotic. The method of synchronizing the dynamics of the target system can be viewed as another method of controlling the chaotic behavior of the target system to the desired orbit. In this type of control, unlike that of the OGY method, the initial values of the system and the control parameter do not need to start near their final values [1].

The controlling scheme of the coupled map is based only on the intrinsic properties of the signal generated by both systems. This is done by comparing the estimated Liapunov exponents of the target map to that of the controlling map. This makes the method very general and applicable to many different systems, possibly with the extension of the method to coupled systems that have only the same class of universality. We also hope to extend this method to systems with many degrees of freedom.

The paper is organized as follows. First we define the method for one dimensional coupled systems (logistic map [13]), and then we generalize it to two-dimensional systems (Chirikov map [14]).

II. DESCRIPTION OF THE METHOD. ONE-DIMENSIONAL CASE

The coupled system is composed of two maps. The controlling map $x^{(1)}$ influences the dynamics of the target map $x^{(2)}$ by the coupling defined by

$$x^{(1)}_{n+1}=f(x^{(1)}_n), \tag{1}$$

$$x^{(2)}_{n+1}=\frac{f(x^{(2)}_n)+\epsilon x^{(1)}_{n+1}}{1+\epsilon}. \tag{2}$$

For the one-dimensional case, for both $x^{(1)}$ and $x^{(2)}$, the logistic map was chosen:

$$f(x)=rx(1-x). \tag{3}$$

The parameter $r^{(1)}$ in the control system was a constant arbitrary value. The parameter $r^{(2)}$ in the target map was changed with subsequent iterations of the map.

The changes in $r^{(2)}$ were determined from the approximate Liapunov exponents, which for the one-dimensional case are defined as

$$\lambda=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\ln\left|\frac{df}{dx}(x_i)\right|. \tag{4}$$

We define the approximate Liapunov for both maps as a running average:

$$\lambda_{n+1}=\frac{\lambda_n+\alpha\ln\left|\frac{df}{dx}(x_{n+1})\right|}{1+\alpha}. \tag{5}$$

The Liapunov exponent $\lambda^{(2)}$ for the target system is calculated from the dynamics of the uncoupled target map:

$$\tilde{x}^{(2)}_{n+1}=r^{(2)}_n x^{(2)}_n(1-x^{(2)}_n). \tag{6}$$

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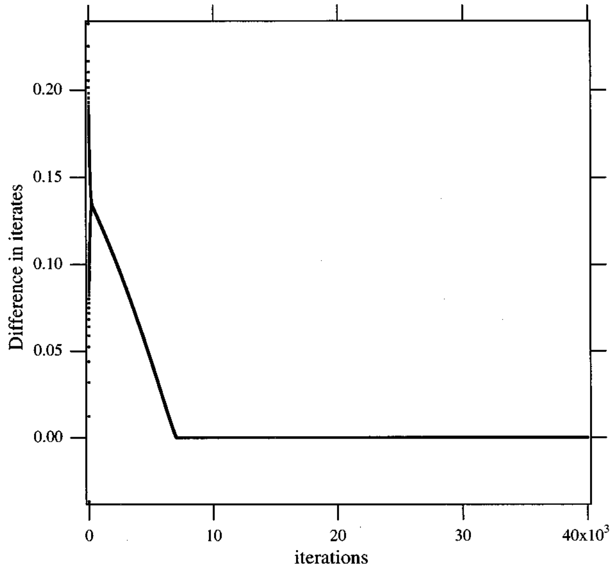

FIG. 1. The difference of the iterates $(x_n^{(1)} - x_n^{(2)})$ of the control and target systems plotted as functions of consecutive iterations for coupled logistic maps. At the beginning of the simulation $r^{(2)} = 3.7$ (chaotic regime). The parameter of the control map is $r^{(1)} = 2.3$ (fixed point). The dynamics of the target map changes rapidly from chaotic dynamics to a fixed point.

We then define a function

$$A(\lambda_n^{(1)}, \lambda_n^{(2)}) = (|\lambda_n^{(1)} - \lambda_n^{(2)}|)^{1/4} \mathrm{sgn}(\lambda_n^{(1)} - \lambda_n^{(2)}). \tag{7}$$

The choice of the definition of the function $A(\lambda_n^{(1)}, \lambda_n^{(2)})$ is to some extent arbitrary. However, the shape of the function was chosen to ensure that very small differences between $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ will be amplified, and larger ones will be dampened to limit possible overshooting. The changes in $r^{(2)}$ are defined as follows:

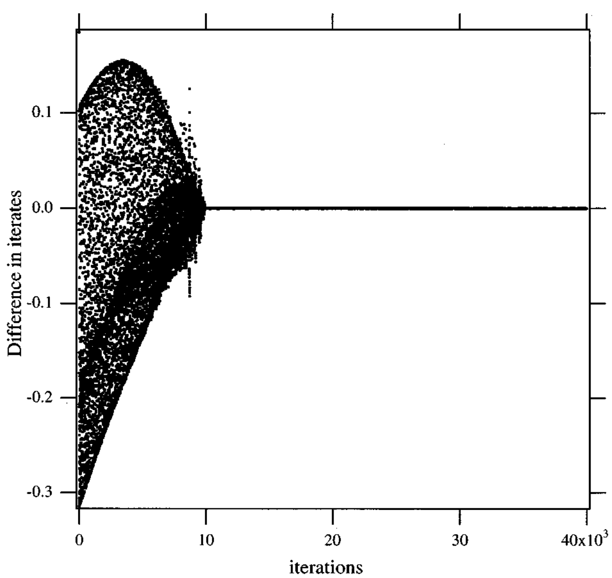

FIG. 2. At the beginning of the simulation $r^{(2)} = 2.1$ (fixed point). The parameter of the control map is $r^{(1)} = 3.71$ (chaotic regime). Both maps synchronize rapidly.

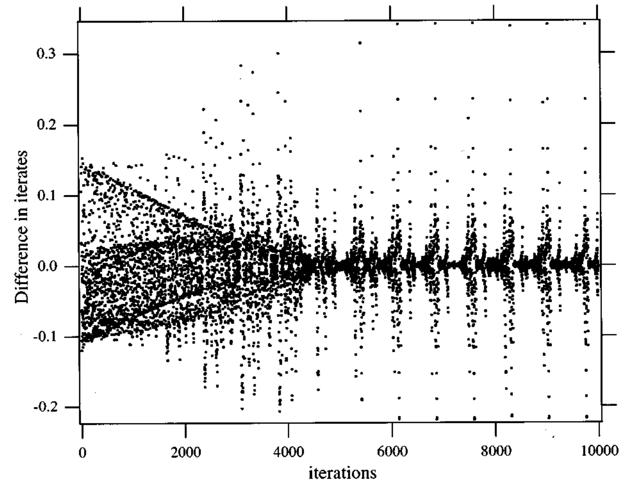

FIG. 3. At high values of $r^{(1)} = 3.75$ the synchronization is intermittent. The target system synchronizes and then escapes into outbursts of uncorrelated activity.

$$r_{n+1}^{(2)} = r_n^{(2)} + \gamma A(\lambda_n^{(1)}, \lambda_n^{(2)}). \tag{8}$$

Figures 1–3 present differences of corresponding iterates $(x_n^{(1)} - x_n^{(2)})$ of the coupled and target maps. Figure 1 shows the case when the dynamics of the control map is a fixed point and the target map starts in the chaotic regime. As one can see both systems synchronize rapidly, and the difference in their iterates tends to zero. Figure 2 shows the opposite case, where the control map is in the chaotic regime and the target map starts in the periodic regime. Figure 3 shows the case where the control is intermittent; that is, the maps synchronize for some well-defined periods and then there are outbursts of nonsynchronized activity. For high values of $r^{(1)} = 3.95$, the control parameter $r^{(2)}$ of the target map does converge to $r^{(1)}$, but the iterates $x_n^{(1)}$ are not synchronized to $x_n^{(2)}$ (Fig. 4).

III. TWO-DIMENSIONAL CASE

The same method of control was then applied to a two-dimensional case. Here the Chirikov map [14] was used:

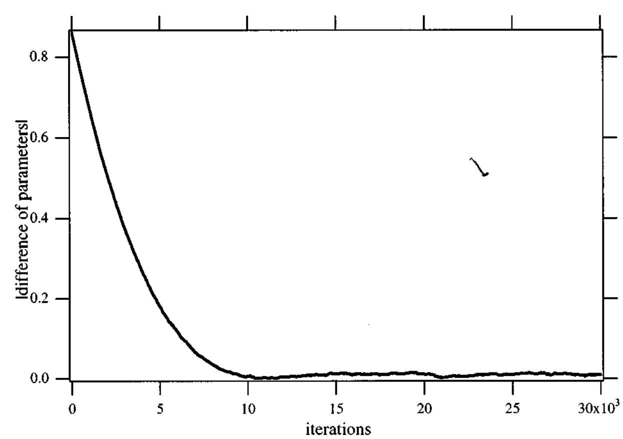

FIG. 4. $r^{(1)} = 3.95$. Even when the target system cannot synchronize its activity to that of the controlling system, value of $r^{(2)}$ converges correctly to $r^{(1)}$. The figure plots $|r^{(2)} - r^{(1)}|$.

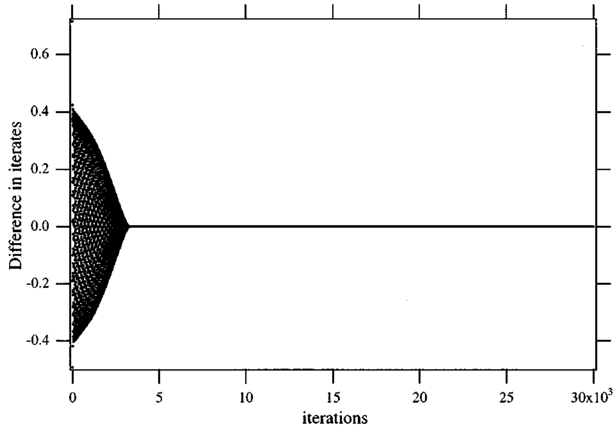

FIG. 5. The difference of iterates $(x_n^{(1)} - x_n^{(2)})$ of coupled systems plotted as functions of consecutive iterations for coupled Chirikov maps at the beginning of the simulation $K^{(2)} = 1.5$. The parameter of the control map is $K^{(1)} = 3.0$.

$$p_{n+1} = p_n - K \sin\theta_n, \tag{9}$$

$$\theta_{n+1} = \theta_n + p_n - K \sin\theta_n, \tag{10}$$

where θ_n and p_n correspond, respectively, to the phase and momentum. The target system is coupled to the control map only through the phase dependence. However, that coupling should be enough to achieve synchronization [5]. The system is defined as follows:

$$\theta_{n+1}^{(1)} = f_\theta(\theta_n^{(1)}, p_n^{(1)}), \tag{11}$$

$$p_{n+1}^{(1)} = f_p(\theta_n^{(1)}, p_n^{(1)}), \tag{12}$$

$$\theta_{n+1}^{(2)} = \frac{f_\theta(\theta_n^{(2)}, p_n^{(2)}, K_n^{(2)}) + \epsilon \theta_{n+1}^{(1)}}{1 + \epsilon}, \tag{13}$$

$$p_{n+1}^{(2)} = f_p(\theta_n^{(2)}, p_n^{(2)}, K_n^{(2)}). \tag{14}$$

For the multidimensional case the Liapunov exponents are defined as [15]

$$(e^{\lambda_1}, \ldots, e^{\lambda_d}) = \lim_{N\to\infty} \left(\text{magnitude of the eigenvalue of } \prod_{n=0}^{N-1} J(\vec{x}_n)^{1/N} \right), \tag{15}$$

where

$$J(\vec{x}) = \left(\frac{\delta G_i}{\delta x_j} \right) \tag{16}$$

is the Jacobian of the matrix of the map $\vec{x}_{n+1} = \vec{G}(\vec{x}_n)$, which for the Chirikov map is defined as

$$J(\theta_n, p_n) = \begin{pmatrix} -K \cos\theta_n & 1 \\ 1 - K \cos\theta_n & 0 \end{pmatrix}. \tag{17}$$

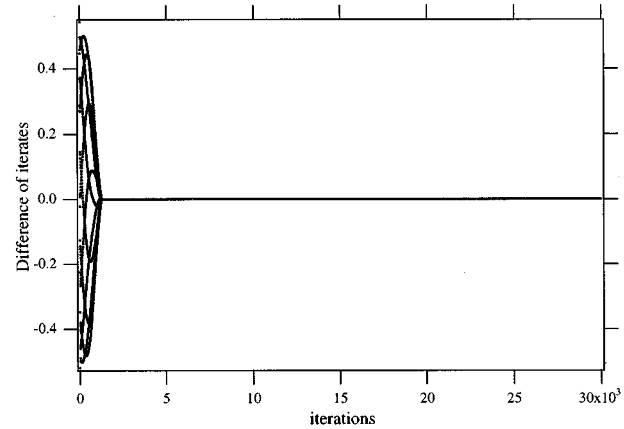

FIG. 6. At the beginning of the simulation $K^{(2)} = 2.5$. The parameter of the control map is $K^{(1)} = 1.1$.

Both Liapunov exponents in both systems were calculated according to the equation

$$\lambda_{n+1}^{(ii)} = \frac{\lambda_n^{(ii)} + \alpha \ln\varphi_n^{(ii)}}{1 + \alpha}, \tag{18}$$

where $\varphi_n^{(ii)}$ are the corresponding eigenvalues of $J(\theta_n, p_n)$.

Then we defined

$$A(\lambda_n^{(11)}, \lambda_n^{(12)}, \lambda_n^{(21)}, \lambda_n^{(22)},) = \max(|\lambda_n^{(11)} - \lambda_n^{(21)}|^{1/4}, |\lambda_n^{(12)} - \lambda_n^{(22)}|^{1/4}). \tag{19}$$

The function A was then multiplied by the appropriate sign to provide the correct direction of the parameter change. The changes in K^1 are defined by Eq. (8).

The results are presented in Figs. 5 and 6 for two different values of the parameter K^1. As in the one-dimensional case, the target map rapidly synchronizes to the control map. The method works well when the Liapunov exponents are a monotonic function of the parameter K.

IV. CONCLUSIONS

In conclusion, we presented an efficient method of controlling the dynamics of the target system based on evaluating the Liapunov exponents for control and target system at every iteration. This method seems to be very robust, and allows a synchronization to any desired orbit from any initial state of the target system. The use of the estimated Liapunov exponents as a feedback control makes the method very general, and possibly it can be extended to coupled systems with the same classes of universality and to systems with many degrees of freedom. Thus it seems to be an advantage over the OGY method of controlling a chaotic dynamics of the system that may find many applications in different branches of science.

ACKNOWLEDGMENT

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[1] E. Ott, C. Grebogi, and J. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[2] R. Lima and M. Pettini, Phys. Rev. A **41**, 726 (1990).
[3] A. Azevedo and S. Pettini, Phys. Rev. Lett. **66**, 1342 (1991).
[4] J. Kowalski, G. Albert, and G. Gross, Phys. Rev. A **42**, 6260 (1990).
[5] L. Kocarec and U. Parlitz, Phys. Rev. Lett. **77**, 2206 (1996).
[6] J. A. J. E. Moreira, and F. W. S. Lima, Phys. Rev. E **52**, R2129 (1995).
[7] D. Hansel and H. Sompolinsky, Phys. Rev. Lett. **68**, 718 (1992).
[8] D. Hansel and H. Sompolinsky, J. Comp. Neurosci. **3**, 7 (1996).
[9] H. Winful and L. Rahman, Phys. Rev. Lett. **65**, 1575 (1990).
[10] B. Huberman and E. Lumer, IEEE Trans. Circuits Syst. **37**, 547 (1990).
[11] S. Sinha, Phys. Lett. A **156**, 475 (1991).
[12] S. Sinha and R. Ramaswamy, Physica D **43**, 118 (1990).
[13] R. May, Nature (London) **261**, 459 (1977).
[14] B. Chirikov, Phys. Rep. **52**, 463 (1979).
[15] H. G. Schuster, *Deterministic Chaos* (Physik-Verlag GmbH, Weinheim, 1984).